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Instabilities of magnetically charged black holes

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Abstract

The stability of the magnetically charged Reissner-Nordström black hole solution is investigated in the context of a theory with massive charged vector mesons. By exploiting the spherical symmetry of the problem, the linear perturbations about the Reissner-Nordström solution can be decomposed into modes of definite angular momentum J. For each value of J, unstable modes appear if the horizon radius is less than a critical value that depends on the vector meson gyromagnetic ratio g and the monopole magnetic charge q/e. It is shown that such a critical radius exists (except in the anomalous case $q = \frac{1}{2}$ with $0 \leq g \leq 2$), provided only that the vector meson mass is not too close to the Planck mass. The value of the critical radius is determined numerically for a number of values of J. The instabilities found here imply the existence of stable solutions with nonzero vector fields ("hair") outside the horizon; unless $q = 1$ and $g > 0$, these will not be spherically symmetric.

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1. Introduction

The Reissner-Nordström solution of the coupled Maxwell-Einstein equations describes a spherically symmetric black hole endowed with electric or magnetic charge. It may remain a solution when the Maxwell theory is embedded in a larger theory. In particular, a trivial extension of the Reissner-Nordström solution gives a magnetically charged solution to a spontaneously broken $SU(2)$ gauge theory. However, it has been shown [1] that if the black hole horizon is sufficiently small (roughly, less than the Compton wavelength of the massive vector meson) this solution develops a classical instability. This instability suggests that there should be some other, stable solution with the same mass and magnetic charge. This solution has been found [2]; it is a gravitationally perturbed magnetic monopole with a black hole inside it.

The spontaneously broken gauge theory can be viewed as a special case of a much wider class of theories obtained by varying the vector meson magnetic moment and by allowing a less restrictive set of interactions. Although such theories will not in general be renormalizable, they are perfectly sensible classically. The magnetically charged solutions of these theories have been studied [3] both with and without gravity. Without gravity, the solutions are singular unless certain relations between the couplings are satisfied. When gravity is added, there are both Reissner-Nordström-type solutions and, for certain ranges of parameters, magnetically charged solutions with nontrivial matter fields outside the horizon. In this paper we analyze the stability of the Reissner-Nordström solutions against small perturbations. We determine the critical value of the horizon radius below which the Reissner-Nordström solution is unstable, and for which the less trivial new solutions are guaranteed to exist. In a future paper we will show how the results of the present analysis can be used to construct such new solutions.

The Reissner-Nordström metric may be written in the form

$$ds^2 = -B(r)dt^2 + B^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{1.1}$$

where

$$
\begin{aligned}
B(r) &= 1 - \frac{2MG}{r} + \frac{4\pi G(Q_M^2 + Q_E^2)}{r^2} \\
&= \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right),
\end{aligned}
\tag{1.2}
$$

with Q_M and Q_E being the magnetic and electric charges, respectively. We consider here the purely magnetic case, with $Q_E = 0$ and $Q_M = q/e$, where the Dirac quantization condition restricts q to be either an integer or a half-integer. Without loss of generality, we may assume that q is positive. In order that the singularity at $r = 0$ be hidden within a horizon, we assume that M is greater than or equal to the extremal mass $M_{\rm ext} = \sqrt{4\pi}(q/e)M_{\rm Pl}$, and hence that the outer horizon $r_+ \equiv r_H \geq \sqrt{4\pi}(q/e)M_{\rm Pl}^{-1}$. The magnetic charge gives rise to a radial magnetic field

$$
F_{\theta\phi} = \frac{q}{e}\sin\theta. \tag{1.3}
$$

All other components of $F_{\mu\nu}$ vanish.

We want to consider the effect on this solution of introducing a massive charged vector field W_μ. This field might be a non-Abelian gauge field which has acquired a mass through the Higgs mechanism (this was the only case considered in Ref. [1]), but this need not be the case. For our purposes here, we will only need the terms in the action which are quadratic in W. These take the form

$$
\begin{aligned}
S_{\rm quad} = \int d^4x \sqrt{-\det(g_{\mu\nu})} \Bigg[&-\frac{1}{2}\left|D_\mu W_\nu - D_\nu W_\mu\right|^2 - m^2 \left|W_\mu\right|^2 \\
&- \frac{ieg}{4} F^{\mu\nu}\left(W_\mu^* W_\nu - W_\nu^* W_\mu\right)\Bigg].
\end{aligned}
\tag{1.4}
$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength while $D_\mu = \partial_\mu - ieA_\mu$ is the gauge covariant derivative. The W mass, m, might depend on the value of some scalar field ϕ, as in the Higgs case, but this dependence can be ignored here; m is then the value corresponding to the vacuum value of ϕ. The last term shown is a magnetic moment term, with the gyromagnetic factor g arbitrary. The integration

is over the region of spacetime outside the horizon of the black hole; it is this region that is relevant for investigation of black hole stability. We must therefore impose boundary conditions at the horizon, which will be discussed later.

The Reissner-Nordström solution is trivially extended to this theory by setting $W_\mu = 0$ and setting all scalar fields at their vacuum values everywhere. We now want to consider fluctuations about this solution. These can be divided into several decoupled sets, comprising the metric and electromagnetic perturbations, the scalar field perturbations, and the vector field perturbations. The first set has been analyzed in detail [4] and shown not to lead to any instability. Because the scalar fields are assumed to be at their vacuum values, it is easy to show that small perturbations in these fields also cannot give rise to any instability; the proof parallels the proof [5] of the no-hair theorem for a free massive scalar field. The analysis of the remaining perturbations, those in W_μ, is the subject of this paper.

The remainder of this paper is organized as follows. In Sec. 2, we exploit the spherical symmetry of the unperturbed solution by expanding W_μ as a sum of eigenmodes of angular momentum and then decomposing the quadratic action into a sum of terms, each of which involves only modes of a given angular momentum. In Sec. 3, we show how both W_t and W_r can be eliminated, thus reducing the problem to one involving at most two radial functions for each value of the angular momentum. In Sec. 4 we show that instabilities develop if the horizon radius r_H is less than some critical value $r_{\rm cr}$, and obtain some analytic bounds on this critical value. Section 5 describes our numerical results for this critical radius. Some concluding remarks are contained in Sec. 6.

2. Angular momentum decomposition of modes

The spherical symmetry of the underlying solution allows the perturbations to be decomposed into modes of definite angular momentum. Because of the non-trivial transformation properties of the monopole vector potential under rotations (or, more fundamentally, because of the extra angular momentum arising from any charge-monopole pair) one cannot use the ordinary spherical harmonics for this decomposition, but must instead employ monopole spherical harmonics. We begin this section by reviewing the properties of these harmonics. Although our eventual interest is in a curved space-time, for the sake of compactness we adopt here a flat space notation with vectors denoted by bold face.

For a scalar field of charge e, the appropriate spherical harmonics are the Wu-Yang monopole harmonics $Y_{qLM}(\theta,\phi)$ [6], which are eigenfunctions of the orbital angular momentum

$$\mathbf{L} = -i\mathbf{r}\times\mathbf{D} - q\hat{\mathbf{r}} \tag{2.1}$$

obeying

$$\begin{aligned} \mathbf{L}^2 Y_{qLM} &= L(L+1)Y_{qLM} \\ L_z Y_{qLM} &= M Y_{qLM}. \end{aligned} \tag{2.2}$$

The quantum numbers L and M take on the values

$$\begin{aligned} L &= q,\ q+1,\ \ldots \\ M &= -L,\ -L+1,\ \ldots,L. \end{aligned} \tag{2.3}$$

These harmonics form a complete orthonormal set, with

$$\int d\Omega\, Y^*_{qLM} Y_{qL'M'} \equiv \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, Y^*_{qLM} Y_{qL'M'} = \delta_{LL'}\,\delta_{MM'}. \tag{2.4}$$

For spin-one fields one must use monopole vector harmonics, which are eigenfunctions of $\mathbf{J} = \mathbf{L}+\mathbf{S}$. In general, there will be several of these corresponding to

the same values of $\mathbf{J}^2$ and J_z. We will find it convenient to use a set [7], denoted by $\mathbf{C}^{(\lambda)}_{qJM}$, in which these are distinguished by the value of $\hat{\mathbf{r}}\cdot\mathbf{S}$. These obey

$$\begin{aligned}
\mathbf{J}^2\,\mathbf{C}^{(\lambda)}_{qJM} &= J(J+1)\,\mathbf{C}^{(\lambda)}_{qJM} \\
J_z\,\mathbf{C}^{(\lambda)}_{qJM} &= M\,\mathbf{C}^{(\lambda)}_{qJM} \\
\hat{\mathbf{r}}\cdot\mathbf{S}\,\mathbf{C}^{(\lambda)}_{qJM} &= i\hat{\mathbf{r}}\times\mathbf{C}^{(\lambda)}_{qJM} = \lambda\,\mathbf{C}^{(\lambda)}_{qJM}
\end{aligned} \tag{2.5}$$

and are normalized so that

$$\int d\Omega\,\mathbf{C}^{(\lambda)*}_{qJM}\cdot\mathbf{C}^{(\lambda')}_{qJ'M'} = \frac{\delta_{JJ'}\,\delta_{MM'}\,\delta_{\lambda\lambda'}}{r^2}. \tag{2.6}$$

The allowed values for the total angular momentum quantum number J occur in integer steps beginning with $q-1$, unless $q=0$ or $1/2$, in which case the minimum value is $J=q$. As usual, the values of M run from $-J$ to J in integer steps. Generically, there are three sets of harmonics for each J, with $\lambda=1$, 0, and -1. However, some of these are absent if J is equal to q or $q-1$. For $J=q-1$ there is only a single family of harmonics, with $\lambda=1$; these harmonics, which will be of particular importance in our analysis, have the important property that both their covariant curl and their covariant divergence vanish. For $J=q>0$ there are two families of harmonics, with $\lambda=1$ and 0. Finally, for $J=q=0$ (i.e., ordinary vector spherical harmonics), only $\lambda=0$ is present.

The last line of Eq. (2.5) shows that the $\lambda=0$ harmonics must be purely radial; the normalization condition (2.6) then fixes them (up to an arbitrary phase) to be

$$\mathbf{C}^{(0)}_{qJM} = \frac{1}{r}\hat{\mathbf{r}}Y_{qJM}. \tag{2.7}$$

Since, as one can easily show, vector harmonics with different values of λ are orthogonal at each point in space, those with $\lambda=\pm1$ can only have angular components.

We will need the formulas for the covariant curls of the vector harmonics. For writing these, it is useful to define

$$\mathcal{J}^2 = J(J+1) - q^2 \tag{2.8}$$

and

$$k_\pm = \sqrt{\frac{\mathcal{J}^2 \pm q}{2}}. \tag{2.9}$$

The curls can then be written as

$$\begin{aligned} \mathbf{D} \times \mathbf{C}^{(\pm 1)}_{qJM} &= \pm \frac{i k_\pm}{r} \mathbf{C}^{(0)}_{qJM} \\ \mathbf{D} \times \mathbf{C}^{(0)}_{qJM} &= \pm \frac{i}{r} \left[k_+ \mathbf{C}^{(1)}_{qJM} - k_- \mathbf{C}^{(-1)}_{qJM} \right]. \end{aligned} \tag{2.10}$$

(Note that $k_+ = 0$ if $J = q - 1$, implying that the corresponding harmonics are indeed curl-free, as asserted above.)

We will also make use of the identity

$$\mathbf{D} Y_{qJM} = k_+ \mathbf{C}^{(1)}_{qJM} + k_- \mathbf{C}^{(-1)}_{qJM}. \tag{2.11}$$

Let us now proceed to the decomposition of the quadratic action. We begin with a mode expansion of W_μ. Recalling that the $\lambda = 0$ harmonics are purely radial while those with $\lambda = \pm 1$ are purely angular, and using Eq. (2.7), we write

$$\begin{aligned} W_t &= \sum_{J=q}^{\infty} \sum_{M=-J}^{J} a^{JM}(r,t) Y_{qJM} \\ W_r &= \frac{1}{r} \sum_{J=q}^{\infty} \sum_{M=-J}^{J} b^{JM}(r,t) Y_{qJM} \\ W_a &= \sum_{J=q-1}^{\infty} \sum_{M=-J}^{J} f_+^{JM}(r,t) \left[\mathbf{C}^{(1)}_{qJM} \right]_a + \sum_{J=q+1}^{\infty} \sum_{M=-J}^{J} f_-^{JM}(r,t) \left[\mathbf{C}^{(-1)}_{qJM} \right]_a, \end{aligned} \tag{2.12}$$

where an index a represents either θ or ϕ. We now insert these expansions into the action and keep only terms quadratic in W_μ. Using the properties of the harmonics

given above (and keeping track of the additional metric factors arising in our curved space-time) we obtain a sum of terms, each corresponding to definite values of J and M:

$$S_{\text{quad}} = \sum_{J=q-1}^{\infty} \sum_{M=-J}^{J} S_{\text{quad}}^{JM}. \tag{2.13}$$

The action for the lowest angular momentum, $J = q-1$, is particularly simple. As noted above, there is only a single multiplet of vector harmonics, and these have the properties of being purely angular and of having vanishing covariant curl. One finds that

$$S_{\text{quad}}^{(q-1)M} = \int dt \int_{r_H}^{\infty} dr \left\{ \frac{1}{B}|\dot{f}_+|^2 - B|f_+'|^2 - m^2|f_+|^2 + \frac{qg}{2r^2}|f_+|^2 \right\}. \tag{2.14}$$

(Here, and henceforth, we omit the superscripts JM on the various coefficient functions.)

The general case is much more complicated, with

$$\begin{aligned} S_{\text{quad}}^{JM} = \int dt \int_{r_H}^{\infty} dr \Bigg\{ &|\dot{b} - ra'|^2 + \frac{1}{B}\left[\left|\dot{f}_+ - k_+ a\right|^2 + \left|\dot{f}_- - k_- a\right|^2 \right] \\ &- B\left[\left|f_+' - \frac{1}{r}k_+ b\right|^2 + \left|f_-' - \frac{1}{r}k_- b\right|^2 \right] - \frac{1}{r^2}\left|k_+ f_+ - k_- f_-\right|^2 \\ &-m^2\left[|f_+|^2 + |f_-|^2 + B|b|^2 - r^2\frac{1}{B}|a|^2 \right] + \frac{gq}{2r^2}\left[|f_+|^2 - |f_-|^2 \right] \Bigg\}. \end{aligned} \tag{2.15}$$

This simplifies a bit when $J = q$, since the terms containing f_- are then absent. As one would expect, it reduces to Eq. (2.14) if J is set equal to $q-1$ and a, b, and f_- are set equal to zero.

Our task is now to analyze the behavior of perturbations governed by the various S_{quad}^{JM}. As a first step, we will now show that the functions a and b (corresponding to W_t and W_r, respectively) can be eliminated from the analysis.

3. Elimination of W_t and W_r

The time derivatives of W_t do not enter the action (2.15). Hence, it (or $a(r,t)$ in the reduced action (2.15)) is a nondynamical constrained field that can be eliminated. We proceed as follows. If the dynamical fields f_+, f_-, and b are assembled into a vector

$$z = \left(\frac{f_+}{\sqrt{B}}, \frac{f_-}{\sqrt{B}}, b \right)^T, \tag{3.1}$$

the action (2.15) may be written in the form

$$S^{JM}_{\text{quad}} = \int dt \int_{r_H}^{\infty} dr \left[\dot{z}^\dagger \dot{z} + (\dot{z}^\dagger F a + a^* F^\dagger \dot{z}) + a^* G a - z^\dagger H z \right], \tag{3.2}$$

where

$$F = \left(-\frac{k_+}{\sqrt{B}}, -\frac{k_-}{\sqrt{B}}, r\frac{\partial}{\partial r} \right)^T, \tag{3.3}$$

$$\begin{aligned} G &= -\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{B}\left(k_+^2 + k_-^2 + r^2 m^2\right) \\ &= F^\dagger F + B^{-1} r^2 m^2 \end{aligned} \tag{3.4}$$

and H is a 3×3 matrix, the explicit form of which we do not need at the moment. (In going from Eq. (2.15) to Eq. (3.2) we have performed an integration by parts and dropped the surface terms. In the next section we will determine the boundary conditions on the fields; with these boundary conditions, the neglect of the surface terms is justified.)

Variations with respect to $a^*(r,t)$ and $z^\dagger(r,t)$ yield the field equations

$$0 = F^\dagger \dot{z} + G a \tag{3.5}$$

and

$$0 = \ddot{z} + F \dot{a} + H z. \tag{3.6}$$

Using the first of these to solve for a, and then substituting the result into the

second equation, we obtain

$$0 = \left(I - FG^{-1}F^{\dagger} \right) \ddot{z} + Hz. \tag{3.7}$$

Instabilities of the Reissner-Nordström solution correspond to solutions of Eq. (3.7) that have a time dependence of the form $e^{\alpha t}$ with real $\alpha > 0$. If the operator acting on $\ddot{z}$ is positive definite, these solutions will be in one-to-one correspondence with the negative eigenvalues of H. To show that this operator is positive, we define the projection operator

$$P = F(F^{\dagger}F)^{-1}F^{\dagger} \tag{3.8}$$

and write the operator in question in the manifestly positive form

$$I - FG^{-1}F^{\dagger} = (I - P) + PF \left[\frac{1}{F^{\dagger}F} - \frac{1}{F^{\dagger}F + B^{-1}r^2m^2} \right] F^{\dagger}P. \tag{3.9}$$

(Note that, although the unstable modes correspond to the negative eigenvalues of H, the actual value of α is not given by the corresponding eigenvalue unless the unstable mode lies in the subspace spanned by $I - P$. This will not affect our calculations, because we are addressing only the question of the existence of instabilities.)

Our problem is thus reduced to the study of the positivity of the potential energy

$$\tilde{E}^{JM}_{\text{quad}} = \int_{r_H}^{\infty} dr z^{\dagger} Hz, \tag{3.10}$$

which is independent of W_t. Because the radial derivative of b does not enter the action, a further reduction is possible. We write $\tilde{E}^{JM}_{\text{quad}}$ as the sum of two integrals,

one of which contains only f_+ and f_-, and the other of which is a perfect square:

$$
\begin{aligned}
\tilde{E}^{JM}_{\rm quad} = \int_{r_H}^{\infty} dr \, \Bigg\{ & B \left[|f'_+|^2 + |f'_-|^2 - \frac{1}{r^2 m^2 + \mathcal{J}^2} |k_+ f'_+ + k_- f'_-|^2 \right] \\
& + \frac{1}{r^2} \, |k_+ f_+ - k_- f_-|^2 + m^2 \left[|f_+|^2 + |f_-|^2 \right] + \frac{qg}{2r^2} \left[|f_+|^2 - |f_-|^2 \right] \Bigg\} \\
+ \int_{r_H}^{\infty} & dr \, B \left| k_+ f'_+ + k_- f'_- - \frac{(r^2 m^2 + \mathcal{J}^2)}{r} \, b \right|^2 .
\end{aligned}
\tag{3.11}
$$

The second integral on the right is clearly nonnegative. Furthermore, given any $f_+(r)$ and $f_-(r)$, it is always possible to choose b so that this integral vanishes. Hence, a necessary and sufficient condition for instability is that there be configurations of $f_+(r)$ and $f_-(r)$ for which the first integral is negative. We denote this quantity by $E^{JM}_{\rm quad}$, and write

$$
E^{JM}_{\rm quad} = \int_{r_H}^{\infty} dr \left[B \, f'^\dagger K_J f' + f^\dagger \left(m^2 I - \frac{V_J}{r^2} \right) f \right] , \tag{3.12}
$$

where $f \equiv (f_+, f_-)^T$ and we have defined matrices

$$
K_J = I - \frac{1}{r^2 m^2 + \mathcal{J}^2} \begin{pmatrix} k_+^2 & k_+ k_- \\ & \\ k_+ k_- & k_-^2 \end{pmatrix} , \qquad J > q \tag{3.13}
$$

and

$$
V_J = \begin{pmatrix} -k_+^2 + \dfrac{qg}{2} & k_+ k_- \\ & \\ k_+ k_- & -k_-^2 - \dfrac{qg}{2} \end{pmatrix} , \qquad J > q. \tag{3.14}
$$

For $J = q-1$ or q, matters are simplified by the absence of f_-, so that K_J and V_J are numbers rather than matrices. Using the fact that k_+^2 is equal to 0 and q,

respectively, for these two cases, we obtain

$$K_{q-1} = 1, \qquad\qquad V_{q-1} = \frac{gq}{2} \tag{3.15}$$

and

$$K_q = \frac{r^2m^2}{r^2m^2+q}, \qquad\qquad V_q = \frac{(g-2)q}{2}. \tag{3.16}$$

(The results for $J = q - 1$ could, of course, be obtained directly from Eq. (2.14).)

The positivity of K_J will be of importance later. This property is obvious for $J = q - 1$ and $J = q$, while for $J > q$ it follows from the positivity of the two eigenvalues

$$k_1 = 1 \qquad \text{and} \qquad k_2 = \frac{r^2m^2}{r^2m^2+\mathcal{J}^2}. \tag{3.17}$$

4. Existence of instabilities

Our task is now to analyze the positivity of the potential energies $E^{JM}_{\rm quad}$. In doing this, it convenient to transform to a tortoise coordinate x defined by

$$\frac{dx}{dr} = B^{-1}(r). \tag{4.1}$$

Spatial infinity, $r = \infty$, corresponds to $x = \infty$, while the horizon radius r_H corresponds to $x = -\infty$. In terms of this coordinate

$$E^{JM}_{\rm quad} = \int_{-\infty}^{\infty} dx \left[\frac{df^\dagger}{dx} K_J \frac{df}{dx} + f^\dagger U_{eff} f\right], \tag{4.2}$$

where

$$U_{eff} = B(r)\left[m^2 I - \frac{V_J}{r^2}\right]. \tag{4.3}$$

Instabilities correspond to the existence of functions $f(x)$ which for which the energy functional is negative. This is equivalent to the existence of a negative

energy bound state in the spectrum of the Hamiltonian

$$H = -\frac{d}{dx}\left(K_J\frac{d}{dx}\right) + U_{eff}(r(x)). \tag{4.4}$$

Because K is a positive matrix, a necessary condition for the occurrence of an instability is that U_{eff} be negative (or have a negative eigenvalue) in some region outside the horizon. This requires that V_J have a positive eigenvalue a^2 and that the horizon radius r_H be less than a value $r_0(J) = a/m$. For $J = q-1$, this gives the requirements that g be positive and that r_H be less than

$$r_0(q-1) = \sqrt{\frac{gq}{2}}m^{-1}. \tag{4.5}$$

For $J = q$, instability can arise only if $g > 2$, while

$$r_0(q) = \sqrt{\frac{(g-2)q}{2}}m^{-1}. \tag{4.6}$$

If $J > q$ the two eigenvalues of V_J are

$$v_\pm = \frac{1}{2}\left[-\mathcal{J}^2 \pm \sqrt{\mathcal{J}^4 + g(g-2)q^2}\right]. \tag{4.7}$$

If $0 \le g \le 2$, both of these eigenvalues are negative and no instabilities can arise. For values of g outside this range, v_+ is positive and

$$r_0(J) = \frac{1}{2}\left[-\mathcal{J}^2 + \sqrt{\mathcal{J}^4 + g(g-2)q^2}\right]^{1/2} m^{-1}, \qquad J > q. \tag{4.8}$$

Note that for fixed q and g, $r_0(J)$ is a decreasing function of J.

Because of the gradient term in the energy, these conditions by themselves are not sufficient for the existence of an unstable mode. In fact, the horizon radius must be less than a critical value $r_{\rm cr}(J)$ that is somewhat smaller than $r_0(J)$. In

the next section we will describe the numerical determination of $r_{\rm cr}$. Before doing so, we show that an unstable mode will always arise if r_H is sufficiently small, i.e., that there is in fact a positive $r_{\rm cr}$. (Our proof implicitly assumes that r_H is greater than the extremal horizon radius. This is a significant constraint on the existence of $r_{\rm cr}$ only if m is comparable to $M_{\rm Pl}$.) To see that this should not be immediately obvious, consider the flat space analogue of our problem in which $B(r)$ and $K_J(r)$ are both identically equal to unity, and the integration is restricted to a region $r > r_* \geq 0$ with the boundary condition that $f(r_*) = f(\infty) = 0$. From the identity

$$\int_{r_*}^{\infty} dr |f'|^2 = \int_{r_*}^{\infty} dr \left[\left| f' - \frac{f}{2r} \right|^2 + \frac{|f|^2}{4r^2} \right] , \tag{4.9}$$

it is easy to see that the energy functional is positive, no matter how small r_* is, if V (or its smallest eigenvalue) is less than 1/4.

The situation is different in curved space, essentially because the metric factor suppresses the effects of the gradient terms near the horizon. To demonstrate this, let us consider the functional

$$E' = \int_{r_H}^{\infty} dr \left[\tilde{B}|f'|^2 + \left(m^2 - \frac{V}{r^2} \right) |f|^2 \right] , \tag{4.10}$$

with $V > 0$ being equal to the positive eigenvalue of V_J and

$$\tilde{B} = 1 - \frac{r_H}{r}. \tag{4.11}$$

For a given value of r_H, the Reissner-Nordström $B(r)$ is less than $\tilde{B}(r)$ everywhere outside the horizon. From this, together with the fact that $K_J(r) \leq 1$, we see that if the potential energy functional (4.2) is positive, then so must be E'. Conversely, if the Hamiltonian H' that we obtain from E' has a bound state, then so must the H of Eq. (4.4), and hence there will be an instability. Our aim is to show that such a bound state develops if r_H is made sufficiently small (but still positive),

with m and V held fixed. Dimensional analysis shows that this limit is equivalent to holding V and r_H fixed and letting m become arbitrarily small. We will find it more convenient to use this latter method.

Defining a tortoise coordinate y by $dy/dr = \tilde{B}^{-1}(r)$, we have

$$H' = -\frac{d^2}{dy^2} + U(r(y)), \tag{4.12}$$

with

$$U = \tilde{B}\left(m^2 - \frac{V}{r^2}\right). \tag{4.13}$$

Now write

$$U = U_1 + U_2 + U_3, \tag{4.14}$$

where

$$U_1(y) = \begin{cases} U(y), & r_H < r < Nr_H \\ 0, & r > Nr_H \end{cases} \tag{4.15}$$

$$U_2(y) = \begin{cases} U(y), & Nr_H < r < r_0 \\ 0, & \text{otherwise} \end{cases} \tag{4.16}$$

$$U_3(y) = \begin{cases} 0, & r < r_0 \\ U(y), & r > r_0, \end{cases} \tag{4.17}$$

with N being some fixed, large number and $r_0 = \sqrt{V}/m$. It is assumed that m is small enough that $r_0 \gg Nr_H \gg r_H$.

Consider the Hamiltonian H'_0 obtained by replacing $U(y)$ by $U_1(y)$. Because $U_1(y)$ is everywhere negative on the real line, a standard result of elementary quantum mechanics guarantees the existence of at least one negative energy bound state. Let us denote the energy of this state by $E_B = -\alpha^2$, and let $\psi(y)$ be the corresponding normalized wave function. We do not need a detailed expression for E_B, but only the fact that it has a smooth nonzero limit as $m \to 0$. This can be

seen by noting that setting $m = 0$ changes the value of $U_1(y)$ at any point by a fractional amount of at most $m^2(Nr_h)^2/V = (Nr_h/r_0)^2 \ll 1$. For a sufficiently large value of r_0, the range $r > r_0$ will be entirely in the classically forbidden region. Hence, in this region, ψ will be of the form $\psi(y) = Ae^{-\alpha y}$ with A only weakly dependent on m.

An upper bound on the minimum eigenvalue $E_{\min}$ of H' can now be obtained by using ψ as a trial wavefunction:

$$
\begin{aligned}
E_{\min} &\leq E_B + \int_{-\infty}^{\infty} dy (U_2 + U_3)|\psi|^2 \\
&< -\alpha^2 + \int_{y(r_0)}^{\infty} dy U_3 |\psi|^2 \\
&< -\alpha^2 + m^2 \int_{y(r_0)}^{\infty} dy |A|^2 e^{-2\alpha y} \\
&= -\alpha^2 + \frac{m^2}{2\alpha} e^{-2\alpha y(r_0)}.
\end{aligned}
\tag{4.18}
$$

For sufficiently small m (i.e., large r_0 and $y(r_0)$), the second term is smaller in magnitude than the first. Hence $E_{\min}$ is negative and we have proven our result.

5. Numerical Results

In the original coordinates, we have an energy functional (3.12) for the perturbations. As we have shown, the black hole is unstable if and only if there exist functions f_+ and f_- such that the appropriate functional is negative. We find such functions by looking for solutions to the eigenvalue equation with negative eigenvalue for the Hamiltonian defined by the functional.[1] The critical radius $r_{\rm cr}$ is the value of the horizon radius above which there are no negative eigenvalues and

1 This equation can also be found by varying the energy functional with respect to f and inserting a Lagrange multiplier to implement the normalization constraint.

below which there is at least one. When inspecting the equations, it is simpler to consider the case where r_H, and thus the boundary condition, is fixed and m varies. (The relevant dimensionless parameter is mr_H.) It is then clear that decreasing m strictly and smoothly decreases the energy, and thus the energy eigenvalues; the critical value of mr_H will be where the lowest eigenvalue crosses from positive to negative.

A critical point in the parameter space, then, will be one for which the Hamiltonian has (generically) one zero eigenvalue and no negative eigenvalues. In general, the wavefunction has two components, f_+ and f_-. For $J \leq q$, however, there is only one component and so to find the critical point we search for values of the parameters such that the solution to the eigenvalue equation with eigenvalue zero has the following properties:

1) The solution exponentially decays to zero at $r \to \infty$. We can solve the equation analytically for large r; we choose the solution that is decaying.

2) The solution goes to a finite value at $r \to r_H$. This is actually a fairly restrictive condition; a typical solution blows up at the horizon. That this is the correct boundary condition can be seen by looking in tortoise coordinates: In these coordinates a solution with a negative energy eigenvalue decays exponentially as $x \to -\infty$. In the limit that the negative eigenvalue goes to zero, this exponential decay is arbitrarily slow.

3) The solution does not change sign.

Properties (1) and (2) are the appropriate boundary conditions; property (3) ensures that there are no negative eigenvalues. (In one dimension, the ground state wavefunction is the unique energy eigenfunction with no zeroes.)

From the argument in the previous section, we see that there will always be a bound state for sufficiently small r_H, or equivalently, sufficiently small m. Thus we can fix the other parameters and search for the critical value of m for which there is a solution to the differential equation that satisfies (1) through (3). We enforce condition (1) by integrating inward from $r \gg r_0$, choosing initial values

to approximate the decaying exponential we expect at large r. (The precise initial values chosen are not important, since the decaying exponential solution will quickly dominate as we integrate to smaller r.) If the solution changes sign before we reach r_H, we know that m is below the critical value; if the solution diverges as $r \to r_H$ without changing sign, we know that m is above the critical value.[2] We can use this information to quite rapidly and accurately zero in on the critical value of m (or, equivalently, r_H).

For $J = q - 1$, the energy functional is given by

$$E_{\text{quad}}^{(q-1)M} = \int_{r_H}^{\infty} dr \left[B\, |f_+'|^2 + \left(m^2 - \frac{qg}{2r^2} \right) |f_+|^2 \right], \tag{5.1}$$

thus the (zero eigenvalue) equation to be solved is

$$-\frac{d}{dr}(B f_+') + \left(m^2 - \frac{qg}{2r^2} \right) f_+ = 0. \tag{5.2}$$

In this case there is only one other dimensionless parameter, gq. (We make the approximation $r_- = 0$, good for $m \ll M_{\text{Pl}}$.) We have calculated the critical mass for a variety of values of this parameter.

For $J = q$, we have

$$E_{\text{quad}}^{qM} = \int_{r_H}^{\infty} dr \left[B \left(\frac{r^2 m^2}{r^2 m^2 + q} \right) |f_+'|^2 + \left(m^2 - \frac{(g-2)q}{2r^2} \right) |f_+|^2 \right], \tag{5.3}$$

which gives the zero eigenvalue equation

$$-\frac{d}{dr} \left[B \left(\frac{r^2 m^2}{r^2 m^2 + q} \right) f_+' \right] + \left(m^2 - \frac{(g-2)q}{2r^2} \right) f_+ = 0. \tag{5.4}$$

In this equation q appears separately from g; in principle, the critical mass depends on both.

2 The algorithm we used in fact searches for a sign change, but if it does not find one, it tests the value of $df/d\ln(r - r_H)$ very close to r_H. If this is positive, it is assumed that the solution will in fact change sign closer to the horizon than the integrator can proceed, and thus m is still below the critical value.

In the case $J > q$, the wavefunction has two components. This gives a two-component equation, which does not have a simple relation between nodes and ground states and which would require tuning of the relative weights of the two components, as well as the mass. In this case, we are able to calculate bounds on the energy by minimizing one-component functionals; these energy bounds give bounds on the critical parameter.

The energy functional for $J > q$ is given by Eq. (3.12). Consider a basis in which V_J is diagonal, and where the upper left element is v_+, the only positive eigenvalue of V_J. (See Eq. (4.7)) The matrix K_J has two positive eigenvalues, k_1 and k_2 (Eq. (3.17)); k_1 is the larger of the two eigenvalues.

Now, we are searching for the minimum energy configuration for the functional $E[f]$ (subject to some normalization constraint). We call this configuration f_0. Now let f_1 be a configuration which minimizes the functional, subject to the additional constraint that the lower component of f is zero (in this particular basis).[3] Clearly $E[f_1] \geq E[f_0]$. Now consider the modified functional $E_-[f]$ obtained by substituting the smaller eigenvalue of K_J, k_2 (actually $k_2 I$), for K_J. Clearly $E_-[f] \leq E[f]$ for all f, so if we minimize E_- with a configuration f_2, $E_-[f_2] \leq E[f_0]$. Thus we have bounded $E[f_0]$ above and below. But both $E[f_1]$ and $E_-[f_2]$ can be calculated with a single component functional, which allows us to put bounds on the critical parameters without tuning the two-component equation.

Now if we write

$$K_J = I - \frac{1}{r^2 m^2 + \mathcal{J}^2} C, \tag{5.5}$$

then in the basis where V_J is diagonal (with the positive eigenvalue v_+ at the upper

3 Since the upper left component of V_J is the only one which is positive, the energy will never be negative if we require that the upper component of f is zero.

left),

$$C = \begin{pmatrix} \frac{\mathcal{J}^2}{2} + \frac{1}{4v}[\mathcal{J}^4 + q^2(g-2)] & \frac{1}{4v}q(g-2)\sqrt{\mathcal{J}^4 - q^2} \\ \frac{1}{4v}q(g-2)\sqrt{\mathcal{J}^4 - q^2} & \frac{\mathcal{J}^2}{2} - \frac{1}{4v}[\mathcal{J}^4 + q^2(g-2)] \end{pmatrix}, \tag{5.6}$$

where $v = \sqrt{\mathcal{J}^4 + q^2 g(g-2)}$. So, to calculate the minimum energy with the lower component constrained to be zero, $E[f_1]$, we minimize the functional

$$E^{JM}_{\text{quad}}[f_u] = \int_{r_H}^{\infty} dr \left[B\left(1 - \frac{1}{r^2m^2 + \mathcal{J}^2}C_{uu}\right)|f_u'|^2 + \left(m^2 - \frac{v_+}{r^2}\right)|f_u|^2 \right], \tag{5.7}$$

where $C_{uu} = \frac{\mathcal{J}^2}{2} + \frac{1}{4v}[\mathcal{J}^4 + q^2(g-2)]$ is the upper left component of the matrix C defined above. If we tune m to solve the zero eigenvalue equation associated with this functional, we obtain a *lower* bound on the critical mass. We bound the energy from below (and thus the critical mass from above) with the E_- described above. Since both V_J and K_J are diagonal in this functional, the upper and lower components of f completely decouple: $E_- = E_-[f_u] + E_-[f_l]$, where f_u, f_l are the upper and lower components of f respectively. Because $E_-[f_l]$ is positive definite, we can set $f_l = 0$ and just use the functional

$$E^{JM}_{-,\text{quad}}[f_u] = \int_{r_H}^{\infty} dr \left[B\left(\frac{r^2m^2}{r^2m^2 + \mathcal{J}^2}\right)|f_u'|^2 + \left(m^2 - \frac{v_+}{r^2}\right)|f_u|^2 \right]. \tag{5.8}$$

Solving the zero eigenvalue equation associated with this functional gives an *upper* bound on the critical mass.

We have calculated and graphed some critical values (or bounds thereon) of mr_H, where m is the W mass and r_H is the outer horizon radius, for varying g, q, and J. (We have calculated these assuming that $r_- = 0$, i.e. a Schwarzschild solution, which will be approximately correct as long as m is far below the Planck scale. Provided that the black hole is not very near extremality, the calculated

values will not be significantly affected by a non-zero r_-. Handling the extremal case is considerably more delicate, and we do not discuss it here.)

First note that the upper and lower bounds on $r_{\rm cr}$ calculated for $J > q$ are quite close, which justifies calculating only these bounds rather than doing the full, two-component calculation. All the curves show the qualitative dependence of the critical radius on g (and J). Most importantly, it appears from our numerical results that $r_{\rm cr}$ is a decreasing function of J; recall that $r_0(J)$ has the same property. Thus, as we reduce the radius of a black hole with fixed charge, it will be the mode with the lowest angular momentum that first becomes unstable. If g is positive and $q > \frac{1}{2}$, then this will always be the mode with $J = q - 1$. If g is negative the $J \leq q$ modes never become unstable (see Eqs. (4.5) and (4.6)) and it is the $J = q + 1$ mode that first becomes unstable. (We do not display results for $g < 0$. For the $J > q$ modes, making the substitution $g \to 2 - g$ does not change the calculated upper bound on $r_{\rm cr}$, although it does change the lower bound. The qualitative behavior is quite similar.) Finally, if $q = \frac{1}{2}$, then there is no instability if $0 \leq g \leq 2$; for $g > 2$, it is the $J = q = \frac{1}{2}$ mode that is the first to become unstable.

It is useful to compare the behavior if $r_{\rm cr}$ with that of r_0. In Figure 2 we plot the ratio of these quantities. Note that for $J \geq q$, we find that $r_{\rm cr} \sim r_0$ for all values of g, with the ratio varying from about $\frac{1}{2}$ to 1. For $J = q - 1$ we find similar behavior for large g, but for very small g, we find that the relationship between g and $r_{\rm cr}$ becomes linear, i.e. that $r_{\rm cr} \propto r_0^2$. (Indeed, for small g, we have $mr_{\rm cr} = (mr_0)^2$, with unit proportionality.) We have no analytical explanation for this behavior. Note also that the values of $r_{\rm cr}/r_0$ for $J = 1$ are very close to the values for $J = 0$, when $g > 2$. Although the differences between these curves are very small, they are much greater than the estimated errors in our calculation.

6. Conclusion

We have investigated the linear stability of magnetically charged black holes in a theory with a massive charged vector field. The Reissner-Nordström solution is stable against small perturbations of the electromagnetic, gravitational, and scalar fields, so linear perturbations of only the W field need be considered. We expanded the W field in angular momentum eigenstates, the monopole vector harmonics. For each value of J, we found the critical horizon radius at which an unstable mode appears. This radius (in units of the W mass) depends upon the angular momentum of the mode, J, the charge of the black hole, q, and the magnetic moment of the W field, g. This makes quantitative the arguments of Ref. [3] concerning the stability of Reissner-Nordström black holes in this theory.

There are some unanswered questions in this analysis. In particular, when the critical horizon radius is close to the radius of an extremal Reissner-Nordström black hole with the same charge, these calculations must be modified. Indeed, if the critical radius we have calculated here is less than the extremal radius, there may be no instability at all. Because of the quite different nature of the coordinates near the extremal horizon, our algorithm for calculating the critical value was ineffective, and so this case was not directly addressed. Another outstanding question is the origin of the linear behavior of the critical radius for $J = q - 1$ near $g = 0$.

In general, it is the mode with the lowest J that becomes unstable at the largest r_H. Consider the evaporation of a Reissner-Nordström black hole with magnetic charge q/e. Once the horizon radius falls below $r_{\rm cr}$ for the lowest mode, we expect that the black hole will be described by a nontrivial classically stable solution with given r_H and q, rather than the unstable Reissner-Nordström solution. If $q = 1$, then the lowest mode has $J = 0$ and is spherically symmetric, so we expect that this new stable solution will also be spherically symmetric. Suppose instead that $q \neq 1$. The lowest mode will have $J > 0$ and the new classically stable solution will not be spherically symmetric. The further evolution of the black hole depenmds on whether or not the theory possesses finite energy magnetic monopoles that are

not black holes. If it does, then as the horizon contracts with further evaporation we expect that the black hole will decay by emitting monopoles and settling down to a $q = 1$ spherically symmetric solution. Eventually the horizon will disappear, leaving behind a simple monopole. If there are no such monopole solutions, the endpoint of the decay will be an asymmetric extremal black hole.

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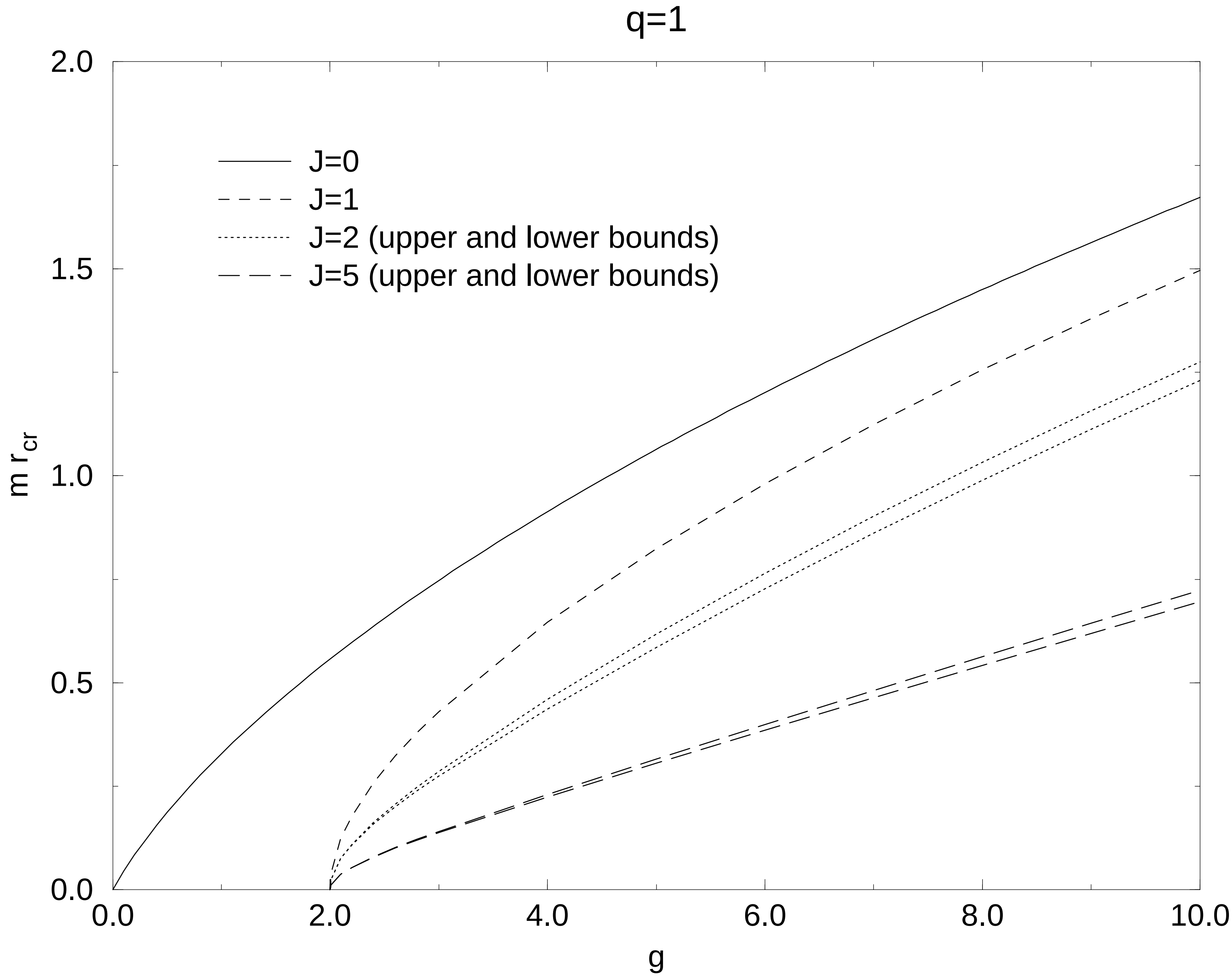
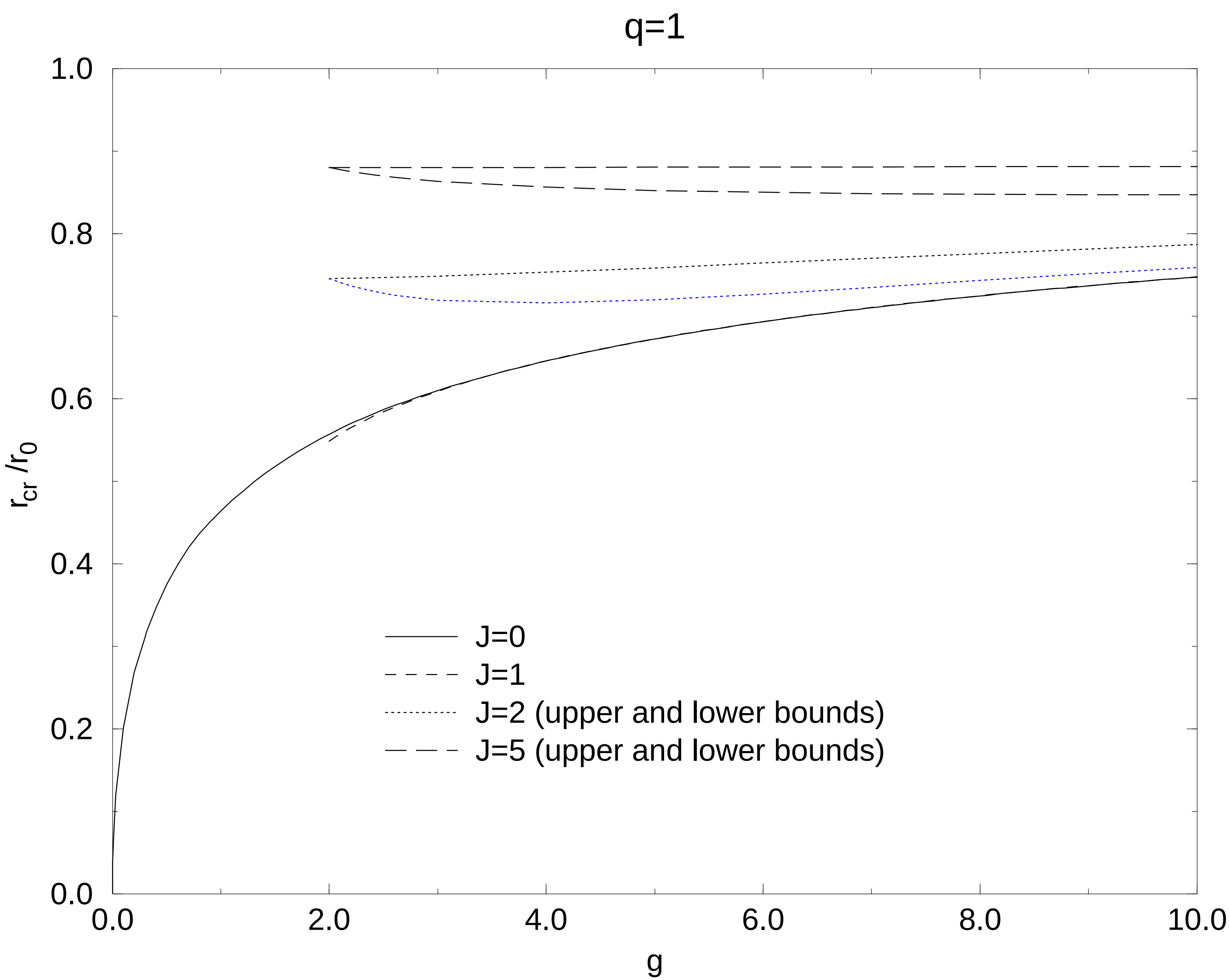
FIGURE CAPTIONS

1) The critical value of mr_H as a function of g for $q = 1$ and $J = 0$, 1, 2, and 5. For $J = 2$ and 5, upper and lower bounds are plotted for $mr_{\rm cr}$, rather than the actual values.

2) The ratio of $r_{\rm cr}$ to r_0 as a function of g for the values of J shown in Figure 1. As in Figure 1, bounds are plotted for $J > q = 1$. Note that the curve for $J = 1$ is largely obscured by the curve for $J = 0$ (for $g > 2$).


q=1
J=0
J=1
J=2 (upper and lower bounds)
J=5 (upper and lower bounds)
m r_cr
g
0.0
0.5
1.0
1.5
2.0
0.0
2.0
4.0
6.0
8.0
10.0


q=1
1.0
0.8
0.6
0.4
0.2
0.0
0.0
2.0
4.0
6.0
8.0
10.0
g
r_{cr}/r_0
J=0
J=1
J=2 (upper and lower bounds)
J=5 (upper and lower bounds)